Harder Simultaneous equations

1. Solve $x^2 - y^2 = 15$ and $x + y = 3$ for x and y.

$$\begin{cases} x^2 - y^2 = 15 \ldots (1) \\ x + y = 3 \ldots (2) \end{cases}$$

Let $x = 1.5 + k$, by (2) $y = 1.5 - k$.

Substitute in (1), $(1.5 + k)^2 - (1.5 + k)^2 = 15$

$\therefore 6k = 15 \Longrightarrow k = 2.5$

$\therefore x = 1.5 + 2.5 = \mathbf{4}$, $y = 1.5 - 2.5 = -\mathbf{1}$.

2. Solve $\begin{cases} (x+1)^2(y+1)^2 = 27xy \\ (x^2+1)(y^2+1) = 10xy \end{cases}$

$$\begin{cases} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 \left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2 = 27 \ldots (1) \\ \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) = 10 \ldots (2) \end{cases}$$

Put $u = \sqrt{x} + \frac{1}{\sqrt{x}}$, $v = \sqrt{y} + \frac{1}{\sqrt{y}}$

Then $x + \frac{1}{x} = u^2 - 2$, $y + \frac{1}{y} = v^2 - 2 \ldots (3)$.

The system of equations become

$$\begin{cases} u^2v^2 = 27 \ldots (4) \\ (u^2 - 2)(v^2 - 2) = 10 \ldots (5) \end{cases}$$

From (5), $u^2v^2 - 2(u^2 + v^2) + 4 = 10$

From (4), $27 - 2(u^2 + v^2) + 4 = 10$

$$u^2 + v^2 = \frac{21}{2} \ldots (6)$$

From (6) and (4), u^2, v^2 are roots of

$$t^2 - \frac{21}{2}t + 27 = 0 \Longrightarrow 2t^2 - 21t + 54 = 0 \Longrightarrow t = 6, \frac{9}{2}$$

$\therefore u^2,\ v^2 = 6, \frac{9}{2} \Longrightarrow u^2 - 2, v^2 - 2 = 4,\ \frac{5}{2} \Longrightarrow x + \frac{1}{x}$, $y + \frac{1}{y} = 4,\ \frac{5}{2}$

(a) $x + \frac{1}{x} = 4 \Longrightarrow x^2 - 4x + 1 = 0 \Longrightarrow x = 2 \pm \sqrt{3}$

(b) $x + \frac{1}{x} = \frac{5}{2} \Longrightarrow 2x^2 - 5x + 2 = 0 \Longrightarrow x = 2, \frac{1}{2}$

Since (1),(2) is symmetric in x, y, the solutions are

$(x, y) = \left(\mathbf{2},\ \mathbf{2 \pm \sqrt{3}}\right), \left(\mathbf{\frac{1}{2}},\ \mathbf{2 \pm \sqrt{3}}\right), \left(\mathbf{2 \pm \sqrt{3}},\ \mathbf{2}\right), \left(\mathbf{2 \pm \sqrt{3}},\ \mathbf{\frac{1}{2}}\right)$.

3. Solve the system, $\begin{cases} ax+by=(x-y)^2 \\ by+cz=(y-z)^2 \\ cz+ax=(z-x)^2 \end{cases}$ where $a,b,c>0$?

$$\begin{cases} ax+by=(x-y)^2 \ldots(1) \\ by+cz=(y-z)^2 \ldots(2) \\ cz+ax=(z-x)^2 \ldots(3) \end{cases}$$

First we solve (1) and (2) by rewriting as:

(i) $ax+by=(x-y)^2 \Longrightarrow y^2-y(2x+b)+(x^2-ax)=0 \ldots(1)$

(ii) $by+cz=(y-z)^2 \Longrightarrow y^2-y(2z+b)+(z^2-cz)=0 \ldots(2)$

$(1)\equiv(2), \begin{cases} 2x+b=2z+b \ldots(4) \\ x^2-ax=z^2-cz \ldots(5) \end{cases}$

From (4), $x=z \ldots(6)$

$(6)\downarrow(5),\ x^2-ax=x^2-cx \Longrightarrow (a-c)x=0 \ldots(7)$

(i) If $a\neq c, x=z=0$, substitute in (1), $y^2-yb=0 \Longrightarrow y=0\ or\ b$

Note that the equation $cz+ax=(z-x)^2 \ldots(3)$ is satisfied for $x=z=0$.

(ii) If $a=c$, from (6), $x=z=t$ is a free variable.

Substitute in (3), $at+at=(t-t)^2 \Longrightarrow t=0$

$(x,y,z)=(0,0,0)$

Similarly we can solve (2), (3) and substitute in (1) or solve (3), (1) and substitute in (2).

Complete solution: $(x,y,z)=(\mathbf{0},\mathbf{0},\mathbf{0}),(\boldsymbol{a},\mathbf{0},\mathbf{0}),(\mathbf{0},\boldsymbol{b},\mathbf{0})(\mathbf{0},\mathbf{0},\boldsymbol{c})$.

4. If $x+3y+5z=200$ and $x+4y+7z=225$, then what is $x+y+z$ equal to?

$$S: \begin{cases} x+3y+5z=200 \ldots(1) \\ x+4y+7z=225 \ldots(2) \\ x+y+z=k \qquad \ldots(3) \end{cases}$$

The coefficient determinant of S is zero.
In order the system to have solution, there must be a free valuable.
Let this be z, and interestingly, z can be any value you like.
For simplicity, put $z=0$.

(1) and (2) becomes $\begin{cases} x+3y=200 \ldots(4) \\ x+4y=225 \ldots(5) \end{cases}$

Solving, $x=125$, $y=25$.
$x+y+z=125+25+0=\mathbf{150}$

5. Solve $\begin{cases} x+\sqrt{y}=7 \\ y+\sqrt{x}=11 \end{cases}$

Method 1

$\begin{cases} x+\sqrt{y}=7 \ldots(1) \\ y+\sqrt{x}=11 \ldots(2) \end{cases}$

Put $u=\sqrt{x},\ v=\sqrt{y}$. The system of equation becomes

$\begin{cases} u^2+v=7 \ \ldots(3) \\ u+v^2=11 \ldots(4) \end{cases}$

From (1), $v=7-u^2 \ldots(5)$

(5) ↓ (4), $u+(7-u^2)^2=11$

$u^4-14u^2+u+38=0$

$\therefore u=2,\ 3.1313, -3.2832, -1.8481.$

Since $u=\sqrt{x}>0$

$\therefore u=2,\ 3.1313 \ldots(6)$

$\therefore x=u^2=4, 9.805$

(6)↓(5), $v=3, -89.14$

Since $v=\sqrt{y}>0, v=3$

$\therefore y=v^2=9$

$\therefore \boldsymbol{x=4, y=9}$ is the only answer.

Method 2 **Numerical method**

$x+\sqrt{y}=7 \Rightarrow x=7-\sqrt{y}$

$y+\sqrt{x}=11 \Rightarrow y=11-\sqrt{x}$

We set up our iterative formula, $\begin{cases} y_n=11-\sqrt{x_n} \ldots(1) \\ x_{n+1}=7-\sqrt{y_n} \ldots(2) \end{cases}$

We choose $x_1=1$, from (1), $y_1=10$.

$\therefore (x_1, y_1)=(1,10)$

From (2), $x_2=7-\sqrt{10}\approx 3.837722$

From (1), $y_1=11-\sqrt{3.837722}\approx 9.0409895$

$\therefore (x_2, y_2)=(3.837722,\ 9.0409895)$

Continue in this way:

$\therefore \boldsymbol{x=4, y=9}$

n	x(n)	y(n)
1	1	10
2	3.83772234	9.040989449
3	3.993176186	9.001706682
4	3.999715567	9.00007111
5	3.999988148	9.000002963
6	3.999999506	9.000000123
7	3.999999979	9.000000005
8	3.999999999	9
9	4	9
10	4	9

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